

ON CHUDNOVSKY-BASED ARITHMETIC ALGORITHMS IN FINITE FIELDS

KEVIN ATIGHEHCHI, STÉPHANE BALLE, ALEXIS BONNECAZE, AND ROBERT ROLLAND

ABSTRACT. Thanks to a new construction of the so-called Chudnovsky-Chudnovsky multiplication algorithm, we design efficient algorithms for both the exponentiation and the multiplication in finite fields. They are tailored to hardware implementation and they allow computations to be parallelized while maintaining a low number of bilinear multiplications. We give an example with the finite field $\mathbb{F}_{16^{13}}$.

1. INTRODUCTION

1.1. Context. Multiplication in finite fields is a fundamental operation in arithmetic and finding efficient multiplication methods remains a topical issue. Let q be a prime power, \mathbb{F}_q the finite field with q elements and \mathbb{F}_{q^n} the degree n extension of \mathbb{F}_q . If $\mathcal{B} = \{e_1, \dots, e_n\}$ is a basis of \mathbb{F}_{q^n} over \mathbb{F}_q then for $x = \sum_{i=1}^n x_i e_i$ and $y = \sum_{i=1}^n y_i e_i$, we have the product

$$(1) \quad z = xy = \sum_{h=1}^n z_h e_h = \sum_{h=1}^n \left(\sum_{i,j=1}^n t_{ijh} x_i x_j \right) e_h,$$

where

$$e_i e_j = \sum_{h=1}^n t_{ijh} e_h,$$

$t_{ijh} \in \mathbb{F}_q$ being some constants. The complexity of a multiplication algorithm in \mathbb{F}_{q^n} depends on the number of multiplications and additions in \mathbb{F}_q . There exist two types of multiplications in \mathbb{F}_q : the scalar multiplication and the bilinear multiplication. The scalar multiplication is the multiplication by a constant (in \mathbb{F}_q) which does not depend on the elements of \mathbb{F}_{q^n} that are multiplied. The bilinear multiplication is a multiplication of elements that depend on the elements of \mathbb{F}_{q^n} that are multiplied. The bilinear complexity is independent of the chosen representation of the finite field. For example, the direct calculation of $z = (z_1, \dots, z_n)$ using (1) requires n^2 non-scalar multiplication $x_i x_j$, n^3 scalar multiplications and $n^3 - n$ additions.

More precisely, the multiplication of two elements of \mathbb{F}_{q^n} is an \mathbb{F}_q -bilinear application from $\mathbb{F}_{q^n} \times \mathbb{F}_{q^n}$ onto \mathbb{F}_{q^n} . Then it can be considered as an \mathbb{F}_q -linear application from the tensor product $\mathbb{F}_{q^n} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ onto \mathbb{F}_{q^n} . Consequently, it can also be considered as an element T of $\mathbb{F}_{q^n}^* \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}^* \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ where \star denotes the dual. When T is written

$$(2) \quad T = \sum_{i=1}^r x_i^* \otimes y_i^* \otimes c_i,$$

where the r elements x_i^* as well as the r elements y_i^* are in the dual $\mathbb{F}_{q^n}^*$ of \mathbb{F}_{q^n} while the r elements c_i are in \mathbb{F}_{q^n} , the following holds for any $x, y \in \mathbb{F}_{q^n}$:

$$x \cdot y = \sum_{i=1}^r x_i^*(x) y_i^*(y) c_i.$$

The decomposition (2) is not unique.

Definition 1.1. Every expression

$$x \cdot y = \sum_{i=1}^r x_i^*(x) y_i^*(y) c_i$$

defines a bilinear multiplication algorithm \mathcal{U} of bilinear complexity $\mu(\mathcal{U}) = r$.

Definition 1.2. The minimal number of summands in a decomposition of the tensor T of the multiplication is called the bilinear complexity of the multiplication and is denoted by $\mu_q(n)$:

$$\mu_q(n) = \min_{\mathcal{U}} \mu(\mathcal{U})$$

where \mathcal{U} is running over all bilinear multiplication algorithms in \mathbb{F}_{q^n} over \mathbb{F}_q .

The bilinear complexity of the multiplication in \mathbb{F}_{q^n} over \mathbb{F}_q has been widely studied. In particular, it was proved in [1] that it is uniformly linear with respect to the degree n of the extension. It follows from a clever algorithm performing multiplication: the so-called multiplication algorithm of Chudnovsky and Chudnovsky. The original Chudnovsky-Chudnovsky algorithm was introduced in 1987 by D.V. and G.V. Chudnovsky [10] and is based on the interpolation on some algebraic curves. From now on, we will denote this algorithm by CCMA.

There is benefit having a low bilinear complexity when considering hardware implementations mainly because it reduces the number of gates in the circuit. In fact, in the so-called non-scalar model (denoted NS), only the bilinear complexity is taken into account and it is assumed that all scalar operations are free. Indeed, this model does not reflect the reality and since the bilinear complexity is not the whole complexity of the algorithm, the complexity of the linear part of the algorithm should also be taken into account. In this paper, we consider two other models. The model S1, which takes into account the number of multiplications without distinguishing between the bilinear ones and the scalar ones. The model S2 which takes into account all operations (multiplications and additions) in \mathbb{F}_q .

Notice that so far, practical implementations of multiplication algorithms over finite fields have failed to simultaneously optimize the number of scalar multiplications, additions and bilinear multiplications.

Regarding exponentiation algorithms, the use of a normal basis is of interest because the q^{th} power of an element is just a cyclic shift of its coordinates. A remaining question is, how to implement multiplication efficiently in order to have simultaneously fast multiplication and fast exponentiation. In 2000, Gao et al. [15] show that fast multiplication methods can be adapted to normal bases constructed with Gauss periods. They show that if \mathbb{F}_{q^n} is represented by a normal basis over \mathbb{F}_q generated by a Gauss period of type (n, k) , the multiplication in \mathbb{F}_{q^n} can be computed with $O(nk \log nk \log \log nk)$ and the exponentiation with $O(n^2 k \log k \log \log nk)$ operations in \mathbb{F}_q (q being small). This result is valuable when k is bounded. However, in the general case k is upper-bounded by $O(n^3 \log^2 nq)$.

In 2009, Couveignes and Lercier construct in [13, Theorem 4] two families of basis (called elliptic and normal elliptic) for finite field extensions from which they obtain a

model Ξ defined as follows. To every couple (q, n) , they associate a model, $\Xi(q, n)$, of the degree n extension of \mathbb{F}_q such that the following holds:

There is a positive constant K such that the following are true:

- Elements in \mathbb{F}_{q^n} are represented by vectors for which the number of components in \mathbb{F}_q is upper bounded by

$$Kn(\log n)^2 \log(\log n)^2.$$

- There exists an algorithm that multiplies two elements at the expense of

$$Kn(\log n)^4 |\log(\log n)|^3$$

multiplications in \mathbb{F}_q .

- Exponentiation by q consists in a circular shift of the coordinates.

Therefore, for each extension of finite field, they show that there exists a model which allows both fast multiplication and fast application of the Frobenius automorphism. Their model has the advantage of existing for all extensions. However, the bilinear complexity of their algorithm is not competitive compared with the best known methods, as pointed out in [13, Section 4.3.4]. Indeed, it is clear that such a model requires at least

$$Kn(\log n)^2 (\log(\log n))^2$$

bilinear multiplications.

Note that throughout the paper, efficiency of algorithms is described in terms of parallel time (depth of the circuit, in number of multiplications), number of processors (width) and total number of multiplications (size). We have $\text{width} \leq \text{size} \leq \text{depth} \cdot \text{width}$.

1.2. New results. We propose another model with the following characteristics:

- Our model is based on CCMA method, thus the multiplication algorithm has a bilinear complexity in $O(n)$, which is optimal.

- Our model is tailored to parallel computation. Hence, the computation time used to perform a multiplication or any exponentiation can easily be reduced with an adequate number of processors. Since our method has a bilinear complexity of multiplication in $O(n)$, it can be parallelized to obtain a constant time complexity using $O(n)$ processors. The previous aforementioned works ([15] and [13]) do not give any parallel algorithm (such an algorithm is more difficult to conceive than a serial one).

- Exponentiation by q is a circular shift of the coordinates and can be considered free. Thus, efficient parallelization can be done when doing exponentiation.

- The scalar complexity of our exponentiation algorithm is reduced compare to a basic exponentiation using CCMA algorithm thanks to a suitable basis representation of the Riemann-Roch space $\mathcal{L}(2D)$ in the second evaluation map. More precisely, the normal basis representation of the residue class field is carried in the associated Riemann-roch space $\mathcal{L}(D)$, and the exponentiation by q consists in a circular shift of the n first coordinates of the vectors lying in the Riemann-Roch space $\mathcal{L}(2D)$.

- Our model uses Coppersmith-Winograd [11] method (denoted CW) or any variants thereof to improve matrix products and to diminish the number of scalar operations. This improvement is particularly efficient for exponentiation.

In term of complexity, we can state the following results, depending on the chosen model (NS, S1 and S2).

Theorem 1.3. *In the non-scalar model NS, there exist multiplication and exponentiation algorithms in \mathbb{F}_{q^n} such that:*

- *Multiplication is done in parallel time in $O(1)$ multiplications in \mathbb{F}_q with $O(n)$ processors, for a total in $O(n)$ multiplications.*
- *Exponentiation is done in parallel time in $O(\log n)$ multiplications in \mathbb{F}_q with $O(n^2/\log^2 n)$ processors, for a total in $O(n^2/\log n)$ multiplications.*

When considering models S1 and S2, two cases can be distinguished for the multiplication complexity. We might be interested either by the complexity of one multiplication or by the average (amortized) complexity of one multiplication when many multiplications are done simultaneously. Regarding exponentiation, a wise use of CW method allows the complexity to be improved.

We can state the followings:

Theorem 1.4. *In the model S1, there exist multiplication and exponentiation algorithms in \mathbb{F}_{q^n} such that:*

- *multiplication:*
 - a) *one multiplication is done in parallel time in $O(1)$ multiplications in \mathbb{F}_q with $O(n^2)$ processors, for a total in $O(n^2)$ multiplications;*
 - b) *in the amortized sense, the parallel time is in $O(1)$ multiplications in \mathbb{F}_q with $O(n^{1+\epsilon})$ processors, for a total in $O(n^{1+\epsilon})$ multiplications where the value of ϵ is approximately 0.38 for the best known matrix product methods;*
- *exponentiation is done in a parallel time of $O(\log n)$ multiplications in \mathbb{F}_q with $O(n^{2+\epsilon}/\log^{2\epsilon} n)$ processors, for a total in $O(n^{2+\epsilon} \log^{1-2\epsilon} n)$ multiplications.*

Theorem 1.5. *In the model S2, there exist multiplication and exponentiation algorithms in \mathbb{F}_{q^n} such that:*

- *multiplication:*
 - a) *one multiplication is done in parallel time in $O(\log n)$ operations in \mathbb{F}_q with $O(n^2/\log n)$ processors, for a total in $O(n^2)$ operations;*
 - b) *in the amortized sense, the parallel time is in $O(\log n)$ operations in \mathbb{F}_q with $O(n^{1+\epsilon}/\log n)$ processors, for a total in $O(n^{1+\epsilon})$ operations; recall that the value of ϵ is approximately 0.38 for the best matrix product methods;*
- *exponentiation is done in a parallel time of $O(\log^2 n)$ operations in \mathbb{F}_q with $O(n^{2+\epsilon}/\log^{1+2\epsilon} n)$ processors, for a total in $O(n^{2+\epsilon} \log^{1-2\epsilon} n)$ operations.*

1.3. Organization of the article. After some background on CCMA algorithm, we describe in Subsection 2.3 our method which leads to an effective algorithm that can directly be implemented. Our algorithm reveals the use of matrix-vector products that can easily be parallelized. In Section 3, we use this algorithm to tackle the problem of computing x^k where $x \in \mathbb{F}_{q^n}$ and $k \geq 1$ and we derive an exponentiation algorithm from the work of von zur Gathen [24, 25]. In Section 4, we focus on the multiplication in $\mathbb{F}_{16^n}/\mathbb{F}_{16}$ and we explain how to construct our algorithm. A Magma [8] implementation of the multiplication algorithm in $\mathbb{F}_{16^{13}}/\mathbb{F}_{16}$ is given in appendix.

2. A NEW APPROACH OF MULTIPLICATION AND EXPONENTIATION ALGORITHMS

First, we present the CCMA algorithm on which is based our method.

2.1. Original algorithm of Chudnovsky-Chudnovsky (CCMA). Let F/\mathbb{F}_q be an algebraic function field over the finite field \mathbb{F}_q of genus $g(F)$. We denote by $N_1(F/\mathbb{F}_q)$ the number of places of degree one of F over \mathbb{F}_q . If D is a divisor, $\mathcal{L}(D)$ denotes the Riemann-Roch space associated to D . We denote by \mathcal{O}_Q the valuation ring of the place Q and by F_Q

its residue class field \mathcal{O}_Q/Q which is isomorphic to $\mathbb{F}_{q^{\deg(Q)}}$ where $\deg(Q)$ is the degree of the place Q . The following theorem that makes effective the original algorithm groups some results of [1].

Theorem 2.1. *Let F/\mathbb{F}_q be an algebraic function field of genus $g(F)$ defined over \mathbb{F}_q and n an integer. Let us suppose that there exists a place Q of degree n .*

Then, if $N_1(F/\mathbb{F}_q) > 2n + 2g - 2$ there is an effective divisor D of degree $n + g - 1$ such that:

- (1) Q is not in the support of D ,
- (2) the evaluation map E defined by

$$\begin{aligned} E : \mathcal{L}(D) &\rightarrow F_Q \\ f &\mapsto f(Q) \end{aligned}$$

is an isomorphism of vector spaces over \mathbb{F}_q ,

- (3) *there exist $2n + g - 1$ places of degree one P_i which are not in the support of D such that the multi-evaluation map T defined by*

$$\begin{aligned} T : \mathcal{L}(2D) &\rightarrow (\mathbb{F}_q)^{2n+g-1} \\ f &\mapsto (f(P_1), \dots, f(P_{2n+g-1})) \end{aligned}$$

is an isomorphism.

The chosen framework is the original CCMA algorithm, namely using only places of degree one and without derivated evaluation (cf. [9]). We transform this algorithm in order that it be adapted to both multiplication and exponentiation computations.

In this context, the construction of this algorithm is based on the choice of the place Q of degree n , the effective divisor D (cf. [2]) and the bases $\mathcal{L}(D)$ and $\mathcal{L}(2D)$.

2.2. Normal bases. Recall some notions on normal bases. The finite field \mathbb{F}_{q^n} will be considered as a vector space of dimension n over the finite field \mathbb{F}_q . Let α be an element of \mathbb{F}_{q^n} such that

$$\left(\alpha, \alpha^q, \alpha^{q^2}, \dots, \alpha^{q^{n-1}} \right)$$

is a basis of \mathbb{F}_{q^n} over \mathbb{F}_q . Such a basis is called a normal basis of \mathbb{F}_{q^n} over \mathbb{F}_q and α is called a cyclic element. Thus, a normal basis is composed of all conjugates of a cyclic element α . There is always a normal basis and furthermore, there is always a primitive normal basis. We call a normal polynomial of degree n over \mathbb{F}_q , a polynomial in $\mathbb{F}_q[X]$, irreducible over \mathbb{F}_q , and having for roots in \mathbb{F}_{q^n} the n conjugates of a cyclic element α . We refer to [20] and [14] for a detailed presentation.

When \mathbb{F}_{q^n} is represented by a normal basis, the q th power of an element is just a cyclic shift of its coordinates. The repeated use of this operation allows exponentiation to be efficiently parallelized. Without normal basis [21], precomputation should be stored for a same base x . This makes sense only when many exponentiation have to be done with this same base and in this case, precomputations are not considered in the running time.

The use of a normal basis has the following benefits:

- Substitute lookup table accesses by circular shifts.
- Reduce prior storage.
- Avoid the constraint of fixing a base.

2.3. Method and strategy of implementation. The construction of the algorithm is based on the choice of the place Q of degree n , the effective divisor D of degree $n + g - 1$ (cf. [2]), the bases of spaces $\mathcal{L}(D)$ and $\mathcal{L}(2D)$ and the basis of the residue class field F_Q of the place Q .

In practice, as in [2], we take as a divisor D one place of degree $n + g - 1$. It has the advantage to solve the problem of the support of divisor D (condition (1) of Theorem 2.1) as well as the problem of the effectivity of the divisor D . Furthermore, we require additional properties.

2.4. Finding places D and Q . To build the good places, we draw them at random and we check that they satisfy the required conditions namely :

- (1) We draw at random an irreducible polynomial $\mathcal{Q}(x)$ of degree n in $\mathbb{F}_q[X]$ and check that this polynomial is :
 - (a) Primitive.
 - (b) Normal.
 - (c) Totally decomposed in the algebraic function field F/\mathbb{F}_q (which implies that there exists a place Q of degree n above the polynomial $\mathcal{Q}(x)$).
- (2) We choose a place Q of degree n among the n places lying above the polynomial $\mathcal{Q}(x)$.
- (3) We draw at random a place D of degree $n + g - 1$ and check that $D - Q$ is a non-special divisor of degree $g - 1$, i.e. $\dim \mathcal{L}(D - Q) = 0$.

Remark 2.2. In practice, it is easy to find Q and D satisfying these properties in our context since there exist many such places. However, it is not true in the general case (it is sufficient to consider an elliptic curve with only one rational point). A sufficient condition for the existence of at least one place of degree n is given by the following inequality:

$$2g(F) + 1 \leq q^{\frac{n-1}{2}} \left(q^{\frac{1}{2}} - 1 \right).$$

When $q \geq 4$, we are sure of the existence of a non-special divisor of degree $g - 1$ [4]. The larger q , the larger the probability to draw a non-special divisor of degree $g - 1$ becomes (Proposition 5.1 [6]) but not necessarily as a difference of two places (this is an open problem). However, in practice, such divisors are easily found.

2.5. Choice of bases of spaces.

2.5.1. The residue field F_Q . When we take a place Q of degree n lying above a normal polynomial in $\mathbb{F}_q[X]$, we mean that the residue class field is the finite field \mathbb{F}_{q^n} for which we choose as a representation basis the normal basis \mathcal{B}_Q generated by a root α of the polynomial $\mathcal{Q}(x)$.

Remark 2.3. Suppose that the context requires the use of a representation basis of \mathbb{F}_{q^n} which is not the basis \mathcal{B}_Q of the residue class field of the place Q . Then, we can easily avoid the problem by a change of basis. This requirement may happen when additional properties on the basis \mathcal{B}_Q (cf. Section 3.2) are required. In particular, it would be the case in our context if we could not find a place Q of degree n above a normal polynomial $\mathcal{Q}(x) \in \mathbb{F}_q[X]$.

2.5.2. The Riemann-Roch space $\mathcal{L}(D)$. As the residue class field F_Q of the place Q is isomorphic to the finite field \mathbb{F}_{q^n} , from now on we identify \mathbb{F}_{q^n} to F_Q . Notice that the choice of D and Q of Section 2.4 are such that the map E of Theorem 2.1 is an isomorphism. Indeed, $\deg(D) = n + g - 1$, $\dim(D - Q) = 0$ yet $\mathcal{L}(D - Q) = \text{Ker}(E)$. In particular,

we choose for basis of $\mathcal{L}(D)$, the reciprocal image \mathcal{B}_D of the basis $\mathcal{B}_Q = (\phi_1, \dots, \phi_n)$ of F_Q by the evaluation map E , namely $\mathcal{B}_D = (E^{-1}(\phi_1), \dots, E^{-1}(\phi_n))$.

2.5.3. The Riemann-Roch space $\mathcal{L}(2D)$. Note that as the divisor D is an effective divisor, we have $\mathcal{L}(D) \subset \mathcal{L}(2D)$. Let P be the map from $\mathcal{L}(2D)$ to $\mathcal{L}(2D)$ defined in the following way: if $f \in \mathcal{L}(2D)$ then $f(Q)$ is in the residue field F_Q of the place Q ; define $P(f) = J \circ E^{-1}(f(Q))$ where J is the injection map from $\mathcal{L}(D)$ into $\mathcal{L}(2D)$. Then P is a linear map from $\mathcal{L}(2D)$ into $\mathcal{L}(2D)$ whose image is $\mathcal{L}(D)$. More precisely, P is a projection from $\mathcal{L}(2D)$ onto $\mathcal{L}(D)$. Let \mathcal{M} be the kernel of P . Then

$$\mathcal{L}(2D) = \mathcal{L}(D) \oplus \mathcal{M}.$$

Remark 2.4. From the definition of P we remark that

- (1) $\mathcal{M} = \{f \in \mathcal{L}(2D) \mid f(Q) = 0\}$,
- (2) for any $f \in \mathcal{L}(2D)$, we have $f(Q) = P(f)(Q)$.

As $\deg 2D > 2g - 2$, the divisor $2D$ is non-special and

$$\dim \mathcal{L}(2D) = 2n + g - 1.$$

Hence, we define as basis of $\mathcal{L}(2D)$, the basis \mathcal{B}_{2D} defined by:

$$\mathcal{B}_{2D} = (f_1, \dots, f_n, f_{n+1}, \dots, f_{2n+g-1})$$

where $\mathcal{B}_M = (f_{n+1}, \dots, f_{2n+g-1})$ is a basis of \mathcal{M} and $\mathcal{B}_D = (f_1, \dots, f_n)$ is the basis of $\mathcal{L}(D)$ defined in Section 2.5.2.

Remark 2.5. As a consequence of the choice of the basis

$$\mathcal{B}_{2D} = (f_1, \dots, f_n, f_{n+1}, \dots, f_{2n+g-1})$$

for $\mathcal{L}(2D)$, if

$$x = (x_1, \dots, x_n, x_{n+1}, \dots, x_{2n+g-1}) \in \mathcal{L}(2D)$$

then

$$P(x) = (x_1, \dots, x_n, 0, \dots, 0).$$

2.6. Product of two elements in \mathbb{F}_{q^n} . In this section, we use as representation basis of spaces $F_Q, \mathcal{L}(D), \mathcal{L}(2D)$, the basis defined in Section 2.5. The product of two elements in \mathbb{F}_{q^n} is computed by the algorithm of Chudnovsky and Chudnovsky. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two elements of \mathbb{F}_{q^n} given by their components over \mathbb{F}_q relative to the chosen basis \mathcal{B}_Q . According to the previous notation, we can consider that x and y are identified to the following elements of $\mathcal{L}(D)$:

$$f_x = \sum_{i=1}^n x_i f_i \quad \text{and} \quad f_y = \sum_{i=1}^n y_i f_i.$$

The product $f_x f_y$ of the two elements f_x and f_y of $\mathcal{L}(D)$ is their product in the valuation ring \mathcal{O}_Q . This product lies in $\mathcal{L}(2D)$. We will consider that x and y are respectively the elements f_x and f_y of $\mathcal{L}(2D)$ where the $n + g - 1$ last components are 0. Now it is clear that knowing x or f_x by their coordinates is the same thing. Let us consider the following Hadamard product in $(\mathbb{F}_q)^{2n+g-1}$:

$$\begin{aligned} & (u_1, \dots, u_{2n+g-1}) \odot (v_1, \dots, v_{2n+g-1}) \\ &= (u_1 v_1, \dots, u_{2n+g-1} v_{2n+g-1}). \end{aligned}$$

Theorem 2.6. *The product of x by y is such that*

$$f_{xy} = P \left(T^{-1} \left(T(f_x) \odot T(f_y) \right) \right).$$

Proof. Indeed, from the definition of T the following holds

$$T(f_x) \odot T(f_y) = T(f_x f_y).$$

Then

$$P \left(T^{-1} \left(T(f_x) \odot T(f_y) \right) \right) = P(f_x f_y).$$

By Remark 2.4 we conclude

$$P(f_x f_y) = f_{xy}.$$

□

We can now present the setup algorithm and the multiplication algorithm. Note that the setup algorithm is only done once.

Algorithm 1 Setup algorithm

INPUT: F/\mathbb{F}_q , Q , D , P_1, \dots, P_{2n+g-1} .

OUTPUT: T and T^{-1} .

- (1) The elements x of the field \mathbb{F}_{q^n} are known by their components relatively to a fixed basis: $x = (x_1, \dots, x_n)$ (where $x_i \in \mathbb{F}_q$).
 - (2) The function field F/\mathbb{F}_q , the place Q , the divisor D and the points P_1, \dots, P_{2n+g-1} are as in Theorem 2.1.
 - (3) Construct a basis $(f_1, \dots, f_n, f_{n+1}, \dots, f_{2n+g-1})$ of $\mathcal{L}(2D)$ where (f_1, \dots, f_n) is the basis of $\mathcal{L}(D)$ defined in section 2.5.2 and $(f_{n+1}, \dots, f_{2n+g-1})$ a basis of \mathcal{M} .
 - (4) Any element $x = (x_1, \dots, x_n)$ in \mathbb{F}_{q^n} is identified to the element $f_x = \sum_{i=1}^n x_i f_i$ of $\mathcal{L}(D)$.
 - (5) Compute the matrices T and T^{-1} .
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Algorithm 2 Multiplication algorithm

INPUT: $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$.

OUTPUT: xy .

- (1) Compute

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \\ z_{n+1} \\ \vdots \\ z_{2n+g-1} \end{pmatrix} = T \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t_1 \\ \vdots \\ t_n \\ t_{n+1} \\ \vdots \\ t_{2n+g-1} \end{pmatrix} = T \begin{pmatrix} y_1 \\ \vdots \\ y_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

- (2) Compute $u = (u_1, \dots, u_{2n+g-1})$ where $u_i = z_i t_i$.
 - (3) Compute $w = (w_1, \dots, w_{2n+g-1}) = T^{-1}(u)$.
 - (4) Return $(xy = (w_1, \dots, w_n))$ (remark that in the previous step we just have to compute the n first components of w).
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In terms of number of multiplications in \mathbb{F}_q , the complexity of this multiplication algorithm is as follows: calculation of z and t needs $2(2n^2 + ng - n)$ multiplications,

calculation of u needs $2n + g - 1$ multiplications and calculation of w needs $2n^2 + ng - n$ multiplications. The total complexity is bounded by $6n^2 + n(3g - 1) + g - 1$.

The asymptotic analysis of our method needs to consider infinite families of algebraic function fields defined over \mathbb{F}_q with increasing genus (or equivalently of algebraic curves) having the required properties. The existence of such families follows from that of recursive towers of algebraic function fields of type Garcia-Stichtenoth [16] reaching the Drinfeld-Vladut bound.

Remark 2.7. Note that, because of the Drinfeld-Vladut bound, in the case of small basis fields \mathbb{F}_2 respectively \mathbb{F}_q with $3 \leq q < 9$, we select the method of the quartic respectively quadratic embedding. In these cases, instead of the embeddings, it would be possible to use places of degree four and respectively two (cf. [7] and [5]) but it requires to generalize our algorithm and it generates significant complications (cf. [23], [3]). Note also that, in the case of the quartic respectively quadratic embedding of the small fields, the operations are precomputed and do not increase the complexity.

Then it is proved [1] from a specialization of the original Chudnovsky algorithm on these families that the bilinear complexity of the multiplication in any degree n extension of \mathbb{F}_q is uniformly linear in q with respect to n . Hence, the number of bilinear multiplications is in $O(n)$ and the genus g of the required curves also necessarily increases in $O(n)$. Consequently, the total number of multiplications/additions/subtractions of our method is in $O(n^2)$ and the total number of bilinear multiplications is in $O(n)$.

On some occasions in the paper, the CW algorithm [11] will be used to decrease the number of scalar operations. Given two square matrices of size n , the product can be computed in $O(n^{2+\epsilon})$ multiplications where $\epsilon < 2.38$. In fact, if we consider a parallel version of the algorithm in the model S1, a product can be performed in $O(1)$ multiplications in \mathbb{F}_q using $O(n^{2+\epsilon})$ processors. This is a consequence of Strassen's normal form theorem [22]. In the model S2 where scalar multiplications and additions/subtractions have same costs, the depth becomes $O(\log n)$ and a rescheduling technique [18] allows the reduction of the width in $O(n^{2+\epsilon}/\log n)$.

Now, we focus on the parallel complexity of our multiplication algorithm. This actually consists of determining the parallel complexity of a constant number of matrix-vector products and the product coordinate-wise of two vectors. First, let us consider the NS model in which a round represents the time interval for a processor to perform one bilinear multiplication in \mathbb{F}_q (scalar operations are considered as free), a multiplication can be carry out in a constant number of rounds with only $O(n)$ processors, as stated in Theorem 1.3.

Regarding the two other models, two cases have to be considered, the non amortized case and the amortized one. In the model S1 (additions and subtractions in \mathbb{F}_q are considered as free), if a round represents the time interval for a processor to perform one multiplication, the product can be performed in a constant number of rounds with $O(n^2)$ processors, as stated in Theorem 1.4a. This width corresponds to the asymptotic number of scalar multiplications. In the model S2 where we take into account all scalar operations in \mathbb{F}_q , a multiplication in \mathbb{F}_{q^n} can be performed in parallel time $O(\log n)$ operations in \mathbb{F}_q using $O(n^2/\log n)$ processors, as stated in Theorem 1.5a.

If we have $\Omega(n)$ multiplications in \mathbb{F}_{q^n} to perform, the width can be decreased (in an amortized sense) by using an optimized matrix multiplication method. It consists in grouping the operands in order that they become the columns of square matrices allowing the use of an efficient method like the CW one. More precisely, the method is as follows:

- (1) Store the $a = \Omega(2n)$ operands (vectors) as columns in square matrices of size $2n + g - 1$, denoted $(B_i)_{i=1 \dots \lceil a/(2n+g-1) \rceil}$;
- (2) Compute the products

$$(TB_i)_{i=1 \dots \lceil a/(2n+g-1) \rceil} = (M_{i,1}, M_{i,2}, M_{i,3}, M_{i,4}, M_{i,5}, \dots),$$

where $M_{i,j}$ represent the columns of TB_i , by using the Coppersmith-Winograd algorithm (or another efficient method);

- (3) Perform the $a/2$ bilinear products $(M_{i,1}, M_{i,2}), (M_{i,3}, M_{i,4}), \dots$;
- (4) Store the $a/2$ results as columns in square matrices, denoted $(B'_i)_{i=1 \dots \lceil a/(4n+2g-2) \rceil}$;
- (5) Compute the products $(T^{-1}B'_i)_{i=1 \dots \lceil a/(4n+2g-2) \rceil}$ with the CW method. The results are then stored in the columns of the resulting matrices.

In the model S1, the necessary width to perform step 3 is in $O(a)$ processors. The overall width is in fact dominated by steps 2 and 5. All these products are performed in constant time using $O(n^{2.38} + an^{1.38})$ processors. This represents, in an amortized sense, $O(n^{1.38})$ processors per multiplication, as stated in Theorem 1.4b.

In the model S2, a multiplication in \mathbb{F}_{q^n} can be performed in parallel time $O(\log n)$ using (thanks to the CW matrix product) $O(n^{1.38}/\log n)$ processors, as stated in Theorem 1.5b. This last result is obtained using a rescheduling technique [18] which allows the parallel computation of the matrix product to be done in $O(\log n)$ operations in \mathbb{F}_q using $O(n^{2.38}/\log n)$ processors, instead of $O(n^{2.38})$ processors without rescheduling [12, 17].

2.7. Product of three elements in \mathbb{F}_{q^n} . The previous algorithm can be iterated in order to obtain the product of three elements x, y, z in $\mathcal{L}(D)$ (or equivalently in \mathbb{F}_{q^n}).

Algorithm 3 Product of three elements

INPUT: $x, y, z \in \mathbb{F}_{q^n}$.

OUTPUT: xyz .

- (1) Compute

$$u = T \circ P \circ T^{-1} (T(x) \odot T(y)),$$

- (2) compute $v = T(z)$,

- (3) then compute $w = u \odot v$ (this is the Hadamard product in $(\mathbb{F}_q)^{2n+g-1}$),

- (4) and finally the result is $P \circ T^{-1}(w)$.
-

In terms of number of multiplications in \mathbb{F}_q , the complexity of this algorithm is as follows: the matrix $T_1 := T \circ P \circ T^{-1}$ can be precomputed and the matrix-vector product needs $(2n + g - 1)^2$ multiplications. The total complexity is then $12n^2 + n(8g - 4) + g^2 - 1$ including $2(2n + g - 1)$ bilinear multiplications (this number is almost doubled compare to the preceding algorithm). Note that the precomputation of T_1 is of interest for the parallel computations. Asymptotically, the complexity is the same as in the previous case.

This algorithm will be used to construct our exponentiation algorithms in Section 3. The form of the involved matrices is analysed in the next subsection.

2.8. Form of the involved matrices. Obviously, the crucial part of the algorithm consuming more time is the iterative call to T_1 , namely the iterative call to the composition

$T \circ P \circ T^{-1}$. The matrix T is

$$\begin{pmatrix} f_1(P_1) & \cdots & f_{2n+g-1}(P_1) \\ f_1(P_2) & \cdots & f_{2n+g-1}(P_2) \\ \vdots & \vdots & \vdots \\ f_1(P_n) & \cdots & f_{2n+g-1}(P_n) \\ \vdots & \vdots & \vdots \\ f_1(P_{2n+g-1}) & \cdots & f_{2n+g-1}(P_{2n+g-1}) \end{pmatrix}.$$

The matrix P is

$$\left(\begin{array}{c|c} I_n & 0 \\ \hline 0 & 0 \end{array} \right)$$

where I_n is the unit matrix of size $n \times n$.

Let us define the following blocks:

$$T = \left(\begin{array}{c|c} U_1 & U_3 \\ \hline U_2 & U_4 \end{array} \right)$$

where U_1 is a $n \times n$ -matrix, U_2 is a $(n+g-1) \times n$ -matrix, U_3 is a $n \times (n+g-1)$ -matrix and U_4 is a $(n+g-1) \times (n+g-1)$ -matrix. In the same way, we can write T^{-1} in the following form:

$$T^{-1} = \left(\begin{array}{c|c} V_1 & V_3 \\ \hline V_2 & V_4 \end{array} \right).$$

Then

$$T_1 = \left(\begin{array}{c|c} U_1 V_1 & U_1 V_3 \\ \hline U_2 V_1 & U_2 V_3 \end{array} \right).$$

Moreover, the following relations hold:

$$\begin{aligned} U_1 V_1 + U_3 V_2 &= I_n, \\ U_1 V_3 + U_3 V_4 &= 0, \\ U_2 V_1 + U_4 V_2 &= 0, \\ U_2 V_3 + U_4 V_4 &= I_{n+g-1}. \end{aligned}$$

An important problem, which is out of the scope of this paper, is to choose a basis (f_i) of $\mathcal{L}(2D)$ and places of degree one P_i in order to obtain a “simple” matrix T_1 . Indeed, a sparse matrix may reduce the number of multiplications.

2.9. Precomputation and storage of scalar multiplications. Having a matrix T (or T_1), the product Tx for all possible $x = (x_1, \dots, x_n, 0, \dots, 0) \in \mathcal{L}(2D)$ can be efficiently precomputed. We choose an integer l dividing n and we set $k = n/l$. For all i such that $0 < i < k - 1$, we denote by $X_{l,i}$ a variable of the form $(X_1, X_2, \dots, X_n, 0, 0, \dots, 0) \in (\mathbb{F}_q)^{2n+g-1}$, where:

$$X_j = \begin{cases} 0 & \text{if } 1 \leq j \leq i \cdot l \\ x_j & \text{if } i \cdot l + 1 \leq j \leq (i+1)l \\ 0 & \text{if } (i+1)l + 1 \leq j \leq n. \end{cases}$$

Thus, we have $x = X_{l,0} + X_{l,1} + \dots + X_{l,k-1}$. We can store in a table $Tab_i^T[\cdot]$ the products $TX_{l,i}$ for the q^l possible values of $X_{l,i}$. After having stored all the precomputations in tables $(Tab_i^T[\cdot])_{i=0 \dots k-1}$, we can compute Ty by evaluating $Tab_0^T[Y_{l,0}] + Tab_1^T[Y_{l,1}] + \dots + Tab_{k-1}^T[Y_{l,k-1}]$.

As an example, for $\mathbb{F}_{16^{13}}$ we can choose $l = 2$, leading to the storage of 6 tables of 256 values plus one table of 128 values.

3. EXPONENTIATION ALGORITHMS

In this section, our aim is to point out the interest, in terms of parallel running time and number of processors involved, of computing an exponentiation in finite fields (*i.e.* x^k in \mathbb{F}_{q^n} with $k \in \mathbb{N}^*$) based on our model. First, it is natural to consider the well known square and multiply algorithm (with say, the method “right-to-left”). We describe this basic algorithm, showing the use of matrices T , P and T_1 . In a second time, we consider a more advanced algorithm, based on an idea from von zur Gathen [24] that we embed in our model. We show that this algorithm reaches optimal depth in terms of operations in \mathbb{F}_q .

3.1. Exponentiation algorithm with a square and multiply method. Let K be the uni-dimensional array of length $s + 1$ containing the bits of k . This array will be indexed by $i = 0, \dots, s$. More precisely

$$k = \sum_{i=0}^s K[i] 2^i.$$

In order to bind operations we must iterate the use of the operator T_1 . We obtain Algorithm 4:

Within a same loop turn, operations under the condition “if” are not used in the subsequent operations. We consider two sets of processing units P_1 and P_2 , the set P_2 running the operations under the condition “if” while the set P_1 deals with the other calculations. We assume that the amounts of resources of P_1 and P_2 are the same. As an example, let us describe the steps in the calculation of x^{15} . Figure 1 depicts the operations made in parallel. We remark that at each step, the set P_1 or P_2 (or both) perform(s) a vector product

Algorithm 4 Square-and-Multiply algorithm (right-to-left)

INPUT: x, k
OUTPUT: x^k
 $X_0 \leftarrow T(x)$
 $X \leftarrow (1, 1, \dots, 1) \in \{0, 1\}^{2n+g-1}$
for i **from** 0 **to** s **do**
 if $K[i] == 1$ **then**
 $X \leftarrow T_1(X \odot X_0)$
 $X_0 \leftarrow T_1(X_0 \odot X_0)$
 $Y \leftarrow P \circ T^{-1}(X)$
return Y

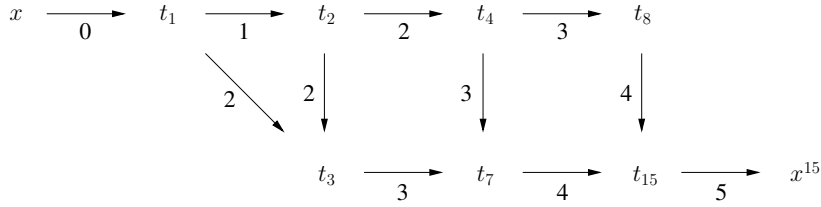


FIGURE 1. Diagram depicting the steps in the calculation of x^{15} . The initial and latest steps are special since they involve only a matrix-vector product (scalar multiplications).

in $(\mathbb{F}_q)^{2n+g-1}$ and subsequently a matrix-vector product, except for the first and the last step for which only a matrix-vector product has to be performed. At step 0, the set P_1 (or P_2) computes $t_1 = T(x)$. The set P_1 computes $t_{2^i} = T_1(t_{i-1} \cdot t_{i-1})$ at step $i = 1 \dots 3$. For its part, the set P_2 computes $t_3 = T_1(t_1 \cdot t_2)$ at step 2, $t_7 = T_1(t_3 \cdot t_4)$ at step 3 and $t_{15} = T_1(t_8 \cdot t_7)$ at step 4. A last step is needed to retrieve the result $x^{15} = P \circ T^{-1}(t_{15})$. Then we can remark that three products in $(\mathbb{F}_q)^{2n+g-1}$ are made in parallel, to which must be added the product performed by P_1 at the step 1, for an overall computation time of four products in $(\mathbb{F}_q)^{2n+g-1}$.

We now consider operations over \mathbb{F}_q and the amount of processing units used. In the model NS, the computation time is in $O(n)$ with only $O(n)$ processors. In the model S1, the computation time is in $O(n)$ with $O(n^2)$ processors and in the model S2, the computation time is in $O(n \log n)$ with $O(n^2 / \log n)$ processors.

3.2. Exponentiation algorithm based on our model. With the use of a normal basis, the base x does not have to be fixed anymore. Moreover, the number of multiplications in \mathbb{F}_{q^n} is reduced and it becomes possible to obtain an "overall" parallel time in $O(\log n)$ without prior storage.

To make use of normal bases, let us now consider the q -ary representation K of the exponent (composed of n terms according to Fermat's Little Theorem) by writing

$$k = \sum_{i=0}^{n-1} K[i]q^i = \sum_{i=0}^{n-1} \sum_{j=0}^t K[i, j]2^j q^i,$$

where $K[i, j]$ for $j = 0, \dots, t$ are the bits of $K[i]$. Let σ be the function such that $\sigma(x, i)$ right shifts i times the vector x . Then, we can express x^k as:

$$\begin{aligned} \prod_{i=0}^{n-1} x^{K[i]q^i} &= \prod_{i=0}^{n-1} \sigma(x, i)^{K[i]} = \prod_{i=0}^{n-1} \prod_{j=0}^t \sigma(x, i)^{K[i,j]2^j} \\ &= \prod_{i=0}^{n-1} \sigma(x^{K[i]}, i) = \prod_{i=0}^{n-1} \sigma \left(\prod_{j=0}^t x^{K[i,j]2^j}, i \right). \end{aligned}$$

This gives two ways of rewriting the Square and Multiply algorithm. In fact, since we are no longer limited to the use of two (sets of) processors, there exist more efficient algorithms, as shown by von zur Gathen [24, 25]. The idea is that short patterns might occur repeatedly in the q -ary representation of the exponent k . Hence, precomputation of all patterns of a given short length $r \geq 1$ allows the overall cost to be lower. By setting $s = \lceil n/r \rceil$ and writing $k = \sum_{0 \leq i < s} k_i q^{ri}$ with $0 \leq k_i < q^r$ for all i , von zur Gathen obtains optimal depth for an appropriate choice of r . In particular it is shown that this result is reachable with width in $O(n/\log n)$ processors.

Lee et al. [19] introduced a rescheduling technique to reduce the number of processors at the counterpart of near-optimal depth. In what follows, we describe our own variant of von zur Gathen algorithm, achieving the same asymptotical efficiencies than the one from Lee et al., while being simpler. We set $r = \lceil \log_q^2 n - 2 \log_q(n) \log_q \log_q n \rceil$ and $u = \lfloor \log_q n - 2 \log_q \log_q n \rfloor$, and rewrite the exponent in the following form:

$$k = \sum_{i=0}^{s-1} \left(\sum_{j=0}^{t-1} K_{i,j} q^{uj} \right) q^{ri}.$$

Thus, $s = \lceil n/r \rceil$ is in $O(n/\log^2 n)$ and $t = \lceil r/u \rceil$ is in $O(\log n)$. The algorithm is divided into five steps:

- (1) for $2 \leq l < q^u$, compute x^l ,
- (2) for $0 \leq i < s$ and $0 \leq j < t$, compute $y_{i,j} = \sigma(x^{K_{i,j}}, u^j)$,
- (3) for $0 \leq i < s$, compute $y_i = \prod_{j=0}^{t-1} y_{i,j}$,
- (4) for $0 \leq i < s$, compute $z_i = \sigma(y_i, ri)$,
- (5) return $x^k = \prod_{i=0}^{s-1} z_i$.

We first examine the parallel time complexities in terms of multiplication in \mathbb{F}_{q^n} . We can use a binary tree of multiplications (executed from root to leaves) to compute Step 1 in $\lceil \log_2(q^u - 1) \rceil$ multiplications using $\max\{2^{\lceil \log_2(q^u - 1) \rceil - 2}, q^u - 1 - 2^{\lceil \log_2(q^u - 1) \rceil - 1}\}$ processors. Step 2 is free. In step 3, each y_i for $0 \leq i < s$ can be computed by distinct processors in $t - 1$ multiplications. Step 4 is free. Step 5 can be computed with a binary multiplication tree (executed from leaves to root) in $\lceil \log_2 s \rceil$ multiplications using $\lfloor s/2 \rfloor$ processors. By summing the times of each step, the overall depth can be upper bounded by

$$\begin{aligned} \left\lceil \log_2 \frac{n}{\log_q^2 n} \right\rceil + \left\lceil \frac{\log_q n + 1}{\log_q n - 2 \log_q \log_q n - 1} \right\rceil + \\ \left\lceil \log_q n \right\rceil + \left\lceil \log_2 \left(\frac{n}{\log_q(n/\log_q^2 n) \log_q n} + 1 \right) \right\rceil \end{aligned}$$

for sufficiently large n . Moreover, by using an optimisation from von zur Gathen, the parallel execution time of Step 1 can be reduced to approximately $\log_2 q + \log_2 \log_q n$.

Thus, the overall depth is in $O(\log n)$ multiplications. It can be noticed that at each parallel step, the number of processors involved stays in $O(n/\log^2 n)$. Consequently, this overall depth is achieved with a width in $O(n/\log^2 n)$ processors.

From now on, we scale up the number of processors to optimize the running time in terms of operations in \mathbb{F}_q . When considering this algorithm in our Chudnovsky model, we consider $O(n/\log^2 n)$ sets of $O(n)$ processors if the scalar multiplications in \mathbb{F}_q are considered free. Otherwise, we consider sets of $O(n^2)$ processors.

Algorithm 5 Precomputation (modified von zur Gathen)

Parallel works assigned to the sets of processors $(P_i)_{i=1..l}$
 where $l = \max(2^{\lceil \log_2(q^u-1) \rceil - 2}, q^u - 1 - 2^{\lceil \log_2(q^u-1) \rceil - 1})$

INPUT: x, q

OUTPUT: x^d for $2 \leq d < q^u$

```

 $x_1 \leftarrow x$ 
 $h \leftarrow \lceil \log_2(q^u - 1) \rceil$ 
for all  $i \in \{1, \dots, h-1\}$  do
  for all  $j \in \{2^i, \dots, 2^{i+1} - 1\}$  the set  $P_{j-2^i+1}$  do
     $x_j \leftarrow P \circ T^{-1}(T(x_{\lceil j/2 \rceil}).T(x_{\lfloor j/2 \rfloor}))$ 
  for all  $j \in \{2^h, \dots, q^u - 1\}$  the set  $P_{j-2^{\lceil \log_2(q^u-1) \rceil}+1}$  do
     $x_j \leftarrow P \circ T^{-1}(T(x_{\lceil j/2 \rceil}).T(x_{\lfloor j/2 \rfloor}))$ 
return  $(x_i)_{i \in \{2, \dots, q^u-1\}}$ 

```

In terms of bilinear multiplications in \mathbb{F}_q , Step 1, described in Algorithm 5, is performed in depth $O(\log n)$ and width $O(n^2/\log^2 n)$. Step 2 and 4 consist in the shift of the first n coordinates and are thus considered free. Step 3, the details of which are left to the reader, is performed in depth $O(\log n)$ and width $O(n^2/\log^2 n)$. The last step, which consists in computing $x^k = \prod_{0 \leq i < s} z_i$, is a binary multiplication tree. It is executed for a cost of $O(\log n)$ bilinear multiplications using $O(n^2/\log^2 n)$ processors. Overall, in the model NS, this algorithm is done in depth $O(\log n)$, width $O(n^2/\log^2 n)$, and size $O(n^2/\log n)$, as stated in Theorem 1.3. We let the reader deduce that, *i*) in the model S1, in terms of multiplications in \mathbb{F}_q , this algorithm is done in depth $O(\log n)$, width $O(n^3/\log^2 n)$, and size $O(n^3/\log n)$; *ii*) in the model S2, in terms of any operations in \mathbb{F}_q , this algorithm is done in depth $O(\log^2 n)$, width $O(n^3/\log^3 n)$, and size $O(n^3/\log n)$.

Reducing the number of scalar operations. We have seen that in the scalar model, the high number of processors is due to the high number of scalar operations (in particular the matrix-vector products Tx , T_1x and $T^{-1}x$). At each step of the computation, the number of multiplications in \mathbb{F}_{q^n} done in parallel is in $O(n/\log^2 n)$ in the worst case. Thus, at a step of the computation involving multiple parallel matrix-vector products, instead of performing $O(n/\log^2 n)$ separate matrix-vector products, we write the $O(n/\log^2 n)$ vectors as columns of a matrix B , then complete this matrix with zero columns in order to obtain a square matrix. Now, we have a product of two square matrices that can be performed using the Coppersmith-Winograd method, thus reducing the number of scalar operations. In the S1 model, the Coppersmith-Winograd product can be performed in $O(1)$ multiplications in \mathbb{F}_q using $O(n^{2.38})$ processors. This optimization allows the exponentiation to be done in depth $O(\log n)$, width $O(n^{2.38})$ and size $O(n^{2.38} \log n)$. In the model S2,

the Coppersmith-Winograd product can be performed in $O(\log n)$ operations in \mathbb{F}_q using $O(n^{2.38}/\log n)$ processors. The optimization allows the exponentiation to be done in depth $O(\log^2 n)$, width $O(n^{2.38}/\log n)$ and size $O(n^{2.38} \log n)$.

Remark that the number of zero columns in the matrix B may not be negligible. Thus, instead of filling B with zero columns in order to obtain a square matrix, we can slightly improve the complexity by using the CW method in the following way: Consider a matrix B containing only $\Theta(n/\log^2 n)$ vectors. Let us denote by L the number of columns of B (L is then in $\Theta(n/\log^2 n)$). We partition B in square submatrices of size L so that B is seen as a column $(B_1, B_2, \dots, B_j)^T$ of blocks (with j in $O(\log^2 n)$). In the same way, we partition T_1 in square submatrices of size L so that a row i of blocks of T_1 is represented as $(A_{i1}, A_{i2}, \dots, A_{ij})$. The sum of products $\sum_{k=1}^j A_{ik} B_k$ corresponds to the i -th block of the resulting column of blocks. A product $A_{ik} B_k$ is done using CW in $O((\frac{n}{\log^2 n})^{2.38})$ scalar multiplications. Since we have $O(\log^4 n)$ such products to compute $T_1 B$, the overall number of scalar multiplications is in $O(n^{2.38}/\log^{0.76} n)$. Consequently, in the model S1 the product $T_1 B$ can be computed in $O(1)$ multiplications in \mathbb{F}_q using $O(n^{2.38}/\log^{0.76} n)$ processors, whereas in the model S2, it can be computed in $O(\log n)$ operations in \mathbb{F}_q using $O(n^{2.38}/\log^{1.76} n)$ processors.

We can substitute these last results in the case of the parallel exponentiation to obtain the stated complexities of Theorem 1.4 and Theorem 1.5, with the current exponent for the best optimized matrix product: in the model S1, x^k can be computed in $O(\log n)$ multiplications in \mathbb{F}_q using $O(n^{2.38}/\log^{0.76} n)$ processors for a size of $O(n^{2.38} \log^{0.24} n)$ multiplications in \mathbb{F}_q , whereas in the model S2, x^k can be computed in $O(\log^2 n)$ operations in \mathbb{F}_q using $O(n^{2.38}/\log^{1.76} n)$ processors for a size of $O(n^{2.38} \log^{0.24} n)$ operations in \mathbb{F}_q .

4. MULTIPLICATION IN $\mathbb{F}_{16^n}/\mathbb{F}_{16}$

Set $q = 16$ and $n = 13, 14, 15$. From now on, F/\mathbb{F}_q denotes the algebraic function field associated to the hyper elliptic curve X with plane model $y^2 + y = x^5$, of genus two. This curve has 33 rational points, which is maximal over \mathbb{F}_q according to the Hasse-Weil bound. We represent \mathbb{F}_{16} as the field $\mathbb{F}_2(a) = \mathbb{F}_2[X]/(P(X))$ where $P(X)$ is the irreducible polynomial $P(X) = X^4 + X + 1$ and a denotes a primitive root of $P(X) = X^4 + X + 1$. Let us give the projective coordinates $(x : y : z)$ of rational points of the curve X :

$$\begin{array}{lll}
 P_\infty = (0 : 1 : 0) & P_2 = (0 : 0 : 1) & P_3 = (0 : 1 : 1) \\
 P_4 = (a : a : 1) & P_5 = (a : a^4 : 1) & P_6 = (a^2 : a^2 : 1) \\
 P_7 = (a^2 : a^8 : 1) & P_8 = (a^3 : a^5 : 1) & P_9 = (a^3 : a^{10} : 1) \\
 P_{10} = (a^4 : a : 1) & P_{11} = (a^4 : a^4 : 1) & P_{12} = (a^5 : a^2 : 1) \\
 P_{13} = (a^5 : a^8 : 1) & P_{14} = (a^6 : a^5 : 1) & P_{15} = (a^6 : a^{10} : 1) \\
 P_{16} = (a^7 : a : 1) & P_{17} = (a^7 : a^4 : 1) & P_{18} = (a^8 : a^2 : 1) \\
 P_{19} = (a^8 : a^8 : 1) & P_{20} = (a^9 : a^5 : 1) & P_{21} = (a^9 : a^{10} : 1) \\
 P_{22} = (a^{10} : a : 1) & P_{23} = (a^{10} : a^4 : 1) & P_{24} = (a^{11} : a^2 : 1) \\
 P_{25} = (a^{11} : a^8 : 1) & P_{26} = (a^{12} : a^5 : 1) & P_{27} = (a^{12} : a^{10} : 1) \\
 P_{28} = (a^{13} : a : 1) & P_{29} = (a^{13} : a^4 : 1) & P_{30} = (a^{14} : a^2 : 1) \\
 P_{31} = (a^{14} : a^8 : 1) & P_{32} = (1 : a^5 : 1) & P_{33} = (1 : a^{10} : 1)
 \end{array}$$

4.1. Construction of the required divisors.

4.1.1. *A place Q of degree n .* It is sufficient to take a place Q of degree n in the rational function field $\mathbb{F}_q(x)/\mathbb{F}_q$, which totally splits in F/\mathbb{F}_q . It is equivalent to choose a monic irreducible polynomial $Q(x) \in \mathbb{F}_q[x]$ of degree n such that its roots α_i in \mathbb{F}_{q^n} satisfy $Tr_{\mathbb{F}_2}(\alpha_i^5) = 0$ for $i = 1, \dots, n$ where the map $Tr_{\mathbb{F}_2}$ denotes the classical function Trace over \mathbb{F}_2 by [20, Theorem 2.25]. In fact, it is sufficient to verify that this property is satisfied for only one root since a finite field is Galois. Moreover, in the context of our method, we require that this irreducible polynomial $Q(x)$ corresponds to a normal polynomial (cf. Section 2.5.1).

For example, for the extension $n = 13$, we choose the primitive normal polynomial

$$(3) \quad \begin{aligned} Q(x) = & x^{13} + a^6 x^{12} + a^5 x^{11} + a^{11} x^{10} + x^9 + a^{12} x^8 + \\ & a^7 x^7 + a^7 x^5 + a^2 x^4 + a^{11} x^3 + a^8 x^2 + a^6 x + a^{14}. \end{aligned}$$

Let b be one primitive root of $Q(x)$. It is easy to check that $Tr_{\mathbb{F}_2}(b^5) = 0$, hence the place $(Q(x))$ of $\mathbb{F}_{16}(x)/\mathbb{F}_{16}$ is totally splitted in the algebraic function field F/\mathbb{F}_q , which means that there exist two places of degree n in F/\mathbb{F}_q lying over the place $(Q(x))$ of $\mathbb{F}_{16}(x)/\mathbb{F}_{16}$. For the place Q of degree n in the algebraic function field F/\mathbb{F}_q , we consider one of the two places in F/\mathbb{F}_q lying over the place $(Q(x))$ of $\mathbb{F}_{16}(x)/\mathbb{F}_{16}$, namely the orbit of the $\mathbb{F}_{16^{13}}$ -rational point $\mathcal{P}_{1i} = (\alpha_i, \beta_i : 1)$ where α_i is a root of $Q(x)$ and $\beta_i = a^6 \alpha_i^{12} + a^{13} \alpha_i^{11} + a \alpha_i^{10} + a^{13} \alpha_i^9 + a^8 \alpha_i^8 + a \alpha_i^7 + a^8 \alpha_i^6 + a^9 \alpha_i^5 + a^5 \alpha_i^4 + a^2 \alpha_i^3 + a^8 \alpha_i^2 + a^{13} \alpha_i + a^{13}$ for $i = 1, \dots, 13$. Notice that the second place is given by the conjugated points $\mathcal{P}_{2i} = (\alpha_i : \beta_i + 1 : 1)$ for $i = 1, \dots, 13$.

4.1.2. *A place D of degree $n+g-1$.* For the divisor D of degree $n + g - 1$, we choose a place D of degree 14 according to the method used for the place Q . We consider the orbit of the $\mathbb{F}_{16^{14}}$ -rational point $\mathcal{P}_{1i} = (\gamma_i, \delta_i : 1)$ where γ_i is a root of $\mathcal{D}(x) = x^{14} + a^9 x^{13} + a^6 x^{12} + a^7 x^{11} + a^{11} x^{10} + a^{12} x^9 + a^{10} x^8 + a^6 x^7 + a^7 x^6 + a^{10} x^5 + a^{14} x^4 + x^3 + x^2 + a^3 x + a$ and $\delta_i = a^4 \gamma_i^{12} + a^8 \gamma_i^{11} + a^7 \gamma_i^9 + a^2 \gamma_i^8 + a^3 \gamma_i^7 + a^8 \gamma_i^6 + a^4 \gamma_i^5 + a^{14} \gamma_i^4 + \gamma_i^2 + a^6 \gamma_i + a^3$ for $i = 1, \dots, 14$. Notice that the second place is given by the conjugated points $\mathcal{P}_{2i} = (\gamma_i, \delta_i + 1 : 1)$ for $i = 1, \dots, 14$.

The place Q and the divisor D satisfy the good properties since the dimension of the divisor $D - Q$ is zero which means that the divisor $D - Q$ is non-special of degree $g - 1$.

4.1.3. *The basis of the residue class field F_Q .* We choose as basis of the residue class field F_Q the normal basis \mathcal{B}_Q associated to the place Q obtained in Section 4.1.1.

4.1.4. *The basis of $\mathcal{L}(D)$.* We choose as basis of the Riemann-Roch space $\mathcal{L}(D)$ the basis $\mathcal{B}_D = (f_1, \dots, f_n)$ such that $E(\mathcal{B}_D) = \mathcal{B}_Q$ is a normal basis of F_Q as in Section 2.5.2. Any element f_i of \mathcal{B}_D is such that

$$f_i(x, y) = \frac{f_{i1}(x)y + f_{i2}(x)}{\mathcal{D}(x)},$$

where $f_{i1}, f_{i2} \in \mathbb{F}_{16}[x]$. To simplify, we set $f_i(x, y) = (f_{i1}(x), f_{i2}(x))$. Let us give the elements of \mathcal{B}_D :

$$f_1(x, y) = (a^{13} x^{11} + a^{10} x^{10} + a^3 x^9 + a^{10} x^8 + a^{14} x^7 + a^{11} x^6 + a^8 x^5 + a^{11} x^4 + x^3 + a x^2 + a^{11} x + a^{11}, a^{12} x^{14} + a^{12} x^{13} + a^9 x^{12} + x^{11} + a^8 x^{10} + a^{13} x^9 + a^{12} x^8 + a x^7 +$$

$$a^5x^6 + x^5 + a^{13}x^4 + a^5x^3 + a^{12}x^2 + a^4x),$$

$$f_2(x, y) = (a^{11}x^{11} + a^5x^{10} + x^9 + ax^8 + a^{14}x^7 + a^{11}x^6 + a^2x^5 + a^4x^4 + a^7x^3 + a^7, a^8x^{14} + a^7x^{13} + a^{12}x^{12} + ax^{11} + a^3x^{10} + a^7x^9 + a^{10}x^8 + a^9x^7 + a^{12}x^6 + a^{11}x^5 + a^{13}x^4 + a^{14}x^3 + a^{13}x^2 + a^2x + a^{12}),$$

$$f_3(x, y) = (a^4x^{11} + a^8x^{10} + a^8x^9 + x^8 + a^2x^7 + a^{14}x^6 + a^2x^5 + a^4x^4 + a^{12}x^3 + a^3x^2 + a^4, ax^{14} + a^7x^{13} + a^6x^{12} + ax^{11} + a^9x^{10} + a^{11}x^9 + a^7x^8 + a^3x^7 + a^7x^6 + ax^5 + a^7x^4 + a^{13}x^3 + a^2x^2 + a^8x),$$

$$f_4(x, y) = (a^5x^{11} + a^{13}x^{10} + a^2x^9 + a^8x^8 + a^9x^7 + a^6x^6 + a^2x^4 + a^6x^3 + a^{13}x^2 + a^9x + a^{11}, a^3x^{14} + a^{12}x^{13} + a^5x^{12} + a^6x^{11} + a^{11}x^{10} + a^3x^9 + a^5x^8 + a^2x^7 + a^{11}x^6 + a^2x^5 + a^{11}x^4 + a^7x^3 + ax^2 + a^4x + a^{12}),$$

$$f_5(x, y) = (a^4x^{11} + a^2x^9 + ax^8 + a^{13}x^7 + a^{12}x^6 + x^5 + ax^4 + a^{13}x^3 + a^{14}x^2 + ax + a^6, a^{11}x^{14} + a^{10}x^{13} + x^{12} + a^{12}x^{11} + a^3x^{10} + a^{12}x^9 + a^9x^8 + a^4x^7 + a^{14}x^6 + a^2x^5 + a^{11}x^4 + a^{11}x^3 + a^6x^2 + a^5x + a^3),$$

$$f_6(x, y) = (a^6x^{11} + a^{12}x^{10} + a^{10}x^9 + a^7x^8 + a^8x^7 + a^6x^5 + x^4 + a^{13}x^3 + a^8x^2 + 1, a^{10}x^{14} + a^4x^{13} + x^{12} + a^4x^{11} + a^2x^{10} + a^7x^9 + a^5x^8 + a^{13}x^7 + ax^6 + a^6x^5 + a^9x^4 + a^7x^3 + a^8x^2 + a^2x + a^{11}),$$

$$f_7(x, y) = (x^{11} + a^5x^{10} + ax^9 + ax^8 + a^{10}x^7 + a^{12}x^6 + a^{14}x^5 + a^3x^4 + a^3x^3 + x^2 + a^4x + a^9, a^7x^{14} + a^3x^{13} + a^8x^{12} + a^4x^{11} + a^6x^{10} + a^6x^9 + a^5x^8 + a^3x^7 + a^{13}x^6 + a^6x^5 + a^4x^4 + a^{14}x^3 + a^{11}x^2 + a^{11}),$$

$$f_8(x, y) = (a^6x^{11} + a^6x^{10} + a^{12}x^9 + x^8 + a^4x^6 + ax^5 + a^{11}x^4 + x^3 + ax^2 + a^{13}x + a^3, a^9x^{14} + a^7x^{13} + a^{14}x^{12} + a^9x^{11} + a^7x^{10} + a^8x^9 + a^{13}x^8 + a^{10}x^7 + a^{10}x^5 + a^{10}x^4 + a^9x^3 + a^7x^2 + ax + a),$$

$$f_9(x, y) = (x^{11} + a^{12}x^{10} + a^{13}x^9 + a^{14}x^8 + a^{13}x^6 + a^{14}x^5 + a^4x^4 + a^{11}x^3 + a^2x^2 + x + a^{11}, x^{14} + a^{14}x^{13} + a^5x^{12} + a^5x^{11} + ax^{10} + a^6x^9 + a^3x^8 + a^{11}x^7 + a^{11}x^6 + a^5x^5 + a^{12}x^4 + x^3 + a^{13}x^2 + a^6x + a^4),$$

$$f_{10}(x, y) = (a^5x^{11} + a^5x^{10} + a^4x^9 + ax^8 + a^9x^7 + a^5x^6 + a^5x^5 + a^4x^4 + a^6x^3 + a^{14}x^2 + a^7x + a^{12}, a^2x^{14} + a^7x^{13} + a^{10}x^{11} + a^9x^{10} + a^{14}x^8 + x^7 + x^6 + a^5x^5 + a^{11}x^4 + a^6x^3 + a^{12}x^2 + a^6x + a^{13}),$$

$$f_{11}(x, y) = (a^{14}x^{11} + a^{14}x^{10} + a^{13}x^9 + a^{11}x^8 + a^7x^7 + a^9x^6 + a^{11}x^5 + a^3x^4 + a^6x^3 + a^8x^2 + x + a^9, a^{12}x^{14} + a^4x^{13} + a^5x^{12} + a^5x^{11} + a^{10}x^{10} + a^{10}x^9 + a^{12}x^8 + a^6x^7 + ax^6 + a^2x^4 + a^9x^2 + a^{11}x + a^{10}),$$

$$f_{12}(x, y) = (a^8x^{11} + a^9x^{10} + a^2x^9 + a^3x^8 + ax^7 + a^{14}x^5 + x^4 + a^{11}x^3 + a^3x^2 + ax + a^{12}, a^{13}x^{14} + a^{13}x^{13} + a^8x^{12} + a^{14}x^{11} + a^4x^{10} + a^{11}x^9 + a^3x^8 + a^{10}x^6 + a^6x^5 + a^9x^3 + a^{14}x^2 + a^{10}x + a^3),$$

$$f_{13}(x, y) = (a^{12}x^{11} + a^5x^{10} + x^9 + a^5x^8 + ax^7 + a^{10}x^6 + a^7x^5 + a^5x^4 + a^3x^2 + x + a^4, a^4x^{14} + a^9x^{13} + a^{14}x^{12} + a^7x^{11} + a^9x^{10} + a^5x^8 + a^{14}x^7 + a^{13}x^6 + a^2x^5 + a^5x^4 +$$

$$a^2x^3 + a^8x^2 + ax),$$

4.1.5. *The basis of $\mathcal{L}(2D)$.* Let $\mathcal{B}_{2D} = (g_1, \dots, g_{2n+g-1})$ be a basis of $\mathcal{L}(2D)$ as defined in Section 2.5.3. Since the divisor D is effective, we can complete the basis f of $\mathcal{L}(D)$ in $\mathcal{L}(2D)$. Then any element g_i of g is such that $g_i(x, y) = f_i(x, y)$ for $i = 1, \dots, n$ and $g_i(x, y) = \frac{g_{i1}(x)y + g_{i2}(x)}{\mathcal{D}^2(x)}$ (which we will denote $g_i(x, y) = (g_{i1}, g_{i2})$), with $g_{i1}, g_{i2} \in \mathbb{F}_{16}[x]$ for $i = n+1, \dots, 2n+g-1$. Moreover, we require the completion of $\mathcal{L}(D)$ by the kernel of the map P . Hence, the set $(g_i)_{i=n+1, \dots, 2n+g-1}$ is a basis of the kernel of the restriction map over $\mathcal{L}(2D)$ of the canonical projection from the valuation ring of the place Q in its residue class field F_Q .

$$g_{14}(x, y) = (a^{13}x^{24} + a^7x^{23} + a^2x^{22} + a^7x^{21} + a^{14}x^{20} + a^2x^{19} + a^9x^{18} + a^6x^{17} + a^{10}x^{16} + a^6x^{15} + a^{12}x^{14} + x^{13} + a^{12}x^{12} + a^3x^{11} + a^8x^9 + a^4x^8 + a^8x^7 + a^9x^6 + a^{12}x^5 + a^{14}x^4 + x^3 + a^8x^2 + x + a^9, a^7x^{26} + ax^{25} + a^{10}x^{23} + a^{14}x^{22} + a^4x^{21} + a^8x^{20} + a^7x^{18} + a^9x^{17} + ax^{16} + a^{12}x^{15} + a^{11}x^{14} + a^2x^{13} + a^9x^{12} + a^{10}x^{11} + a^9x^{10} + ax^8 + a^{11}x^7 + a^5x^6 + a^6x^5 + a^5x^4 + a^{14}x^3 + a^5x^2 + a^{11}x + a^8),$$

$$g_{15}(x, y) = (a^{13}x^{25} + a^7x^{24} + a^2x^{23} + a^7x^{22} + a^{14}x^{21} + a^2x^{20} + a^9x^{19} + a^6x^{18} + a^{10}x^{17} + a^6x^{16} + a^{12}x^{15} + x^{14} + a^{12}x^{13} + a^3x^{12} + a^8x^{10} + a^4x^9 + a^8x^8 + a^9x^7 + a^{12}x^6 + a^{14}x^5 + x^4 + a^8x^3 + x^2 + a^9x, a^7x^{27} + ax^{26} + a^{10}x^{24} + a^{14}x^{23} + a^4x^{22} + a^8x^{21} + a^7x^{19} + a^9x^{18} + ax^{17} + a^{12}x^{16} + a^{11}x^{15} + a^2x^{14} + a^9x^{13} + a^{10}x^{12} + a^9x^{11} + ax^9 + a^{11}x^8 + a^5x^7 + a^6x^6 + a^5x^5 + a^{14}x^4 + a^5x^3 + a^{11}x^2 + a^8x),$$

$$g_{16}(x, y) = (a^7x^{25} + a^{10}x^{24} + a^9x^{23} + a^{11}x^{22} + a^4x^{21} + x^{20} + a^5x^{19} + ax^{18} + a^{14}x^{17} + a^6x^{16} + a^4x^{15} + a^5x^{14} + a^9x^{13} + a^{10}x^{12} + a^3x^{11} + a^{14}x^{10} + a^{10}x^9 + a^4x^8 + a^8x^7 + a^{10}x^6 + x^5 + a^5x^4 + a^4x^2 + a^4x + a, ax^{27} + a^4x^{26} + a^{11}x^{25} + a^{13}x^{24} + x^{23} + a^{12}x^{22} + a^8x^{20} + a^{11}x^{19} + a^4x^{18} + ax^{17} + a^8x^{16} + a^8x^{15} + a^2x^{14} + a^9x^{13} + a^{14}x^{12} + a^2x^{11} + a^5x^{10} + a^3x^9 + a^{11}x^8 + a^2x^7 + a^{12}x^6 + a^{10}x^5 + ax^4 + a^9x^3 + a^5x^2 + x + a^7),$$

$$g_{17}(x, y) = (a^{10}x^{25} + a^{10}x^{24} + a^2x^{23} + ax^{21} + a^2x^{20} + a^9x^{19} + a^{13}x^{18} + a^3x^{17} + a^{11}x^{16} + x^{15} + a^{12}x^{14} + a^7x^{12} + ax^{11} + a^{12}x^{10} + a^2x^9 + a^{12}x^7 + a^{10}x^6 + a^5x^5 + a^{13}x^4 + a^{14}x^3 + a^5x^2 + a^{12}x + a^{10}, a^4x^{27} + a^4x^{26} + a^3x^{25} + a^{12}x^{24} + a^3x^{23} + a^3x^{22} + a^8x^{21} + a^3x^{20} + a^{13}x^{19} + a^3x^{18} + a^{11}x^{17} + a^{10}x^{16} + a^{11}x^{15} + a^6x^{14} + ax^{13} + a^{14}x^{12} + a^3x^{11} + a^5x^{10} + a^{10}x^9 + a^7x^8 + ax^6 + a^2x^5 + a^2x^4 + a^3x^3 + a^6x^2 + x + a),$$

$$g_{18}(x, y) = (a^{10}x^{25} + a^5x^{24} + a^{12}x^{23} + a^{14}x^{22} + a^{12}x^{21} + a^{14}x^{20} + x^{19} + a^{11}x^{18} + a^3x^{17} + a^4x^{16} + ax^{15} + a^{11}x^{14} + a^5x^{13} + a^{14}x^{12} + a^3x^{11} + a^3x^{10} + a^{13}x^9 + x^8 + a^7x^7 + a^4x^6 + a^{13}x^5 + a^7x^4 + a^{14}x^3 + x^2 + a^8x + a^{13}, a^4x^{27} + a^9x^{26} + x^{25} + x^{24} + a^8x^{23} + a^{14}x^{22} + a^3x^{21} + a^3x^{20} + x^{19} + x^{18} + a^{13}x^{16} + a^4x^{15} + a^{10}x^{14} + a^{11}x^{13} + a^9x^{12} + a^{12}x^{11} + a^{11}x^{10} + a^{12}x^9 + x^8 + a^4x^7 + a^9x^6 + x^5 + x^4 + a^8x^3 + a^4x^2 + a^{13}x + a^4),$$

$$g_{19}(x, y) = (a^5x^{25} + a^{13}x^{24} + a^5x^{23} + a^2x^{22} + ax^{21} + ax^{20} + ax^{19} + a^{11}x^{18} + a^8x^{17} + a^4x^{15} + a^3x^{14} + a^2x^{13} + a^4x^{12} + a^{12}x^{11} + x^{10} + a^6x^9 + a^8x^8 + a^{12}x^7 + a^3x^6 + a^7x^5 + a^7x^4 + a^{11}x^3 + a^7x^2 + a^{12}x + a^{13}, a^9x^{27} + a^4x^{26} + a^{12}x^{25} + a^6x^{24} + a^2x^{23} + a^2x^{22} + a^3x^{21} + a^2x^{20} + a^3x^{19} + a^{12}x^{18} + a^9x^{17} + a^3x^{15} + a^7x^{14} + a^{13}x^{13} + a^{13}x^{12} + a^{10}x^{11} + a^5x^{10} + a^8x^9 + ax^8 + a^7x^7 + a^{12}x^6 + a^2x^5 + a^6x^4 + a^9x^3 + a^{12}x^2 + a^6x + a^4),$$

$$g_{20}(x, y) = (a^{13}x^{25} + a^3x^{24} + a^{12}x^{23} + a^5x^{22} + a^5x^{21} + a^7x^{20} + a^6x^{19} + a^2x^{18} + x^{17} + a^{13}x^{16} + a^{13}x^{15} + a^3x^{14} + a^5x^{13} + a^7x^{12} + a^{10}x^{11} + a^{11}x^{10} + a^4x^8 + a^9x^7 + a^4x^6 + a^7x^5 + a^4x^2 + a^4x + a^8, a^4x^{28} + a^4x^{27} + x^{26} + a^{11}x^{24} + x^{23} + a^9x^{22} + a^2x^{21} + a^8x^{19} + x^{18} + a^5x^{17} + x^{16} + a^8x^{14} + a^7x^{13} + a^{14}x^{12} + a^6x^{11} + a^{12}x^{10} + a^{13}x^9 + a^6x^8 + a^3x^7 + a^3x^6 + a^2x^5 + a^8x^3 + ax^2 + a^3x + a^{14}),$$

$$g_{21}(x, y) = (a^8x^{25} + a^{13}x^{24} + a^{13}x^{23} + a^2x^{22} + a^{11}x^{21} + ax^{20} + a^5x^{19} + a^3x^{17} + a^{13}x^{16} + x^{15} + ax^{14} + a^2x^{13} + a^3x^{12} + ax^{11} + a^{10}x^{10} + x^9 + a^3x^8 + x^7 + a^{13}x^6 + a^2x^5 + a^{12}x^4 + x^3 + a^9x^2 + ax + a^4, a^2x^{27} + a^7x^{26} + a^{10}x^{25} + a^{14}x^{24} + a^9x^{23} + a^{14}x^{22} + a^6x^{21} + ax^{20} + a^5x^{19} + x^{18} + a^4x^{17} + a^{14}x^{16} + a^9x^{15} + a^8x^{14} + a^8x^{13} + a^{11}x^{12} + a^5x^{11} + a^7x^{10} + a^{12}x^8 + a^2x^7 + a^9x^6 + a^2x^5 + a^8x^4 + a^{14}x^3 + a^2x^2 + a^{13}x + a^{13}),$$

$$g_{22}(x, y) = (a^{13}x^{25} + a^2x^{24} + a^4x^{23} + a^3x^{22} + a^2x^{21} + ax^{20} + a^4x^{19} + a^8x^{17} + a^3x^{16} + a^6x^{15} + a^{11}x^{14} + a^5x^{13} + a^2x^{12} + ax^{11} + ax^{10} + a^5x^9 + a^7x^8 + a^{11}x^7 + a^3x^6 + a^{12}x^5 + a^3x^4 + a^{13}x^3 + a^3x^2 + a^9x + a^{11}, a^7x^{27} + a^{11}x^{26} + a^4x^{25} + a^8x^{24} + a^{10}x^{23} + a^{12}x^{22} + ax^{21} + a^9x^{20} + a^{11}x^{19} + a^2x^{18} + a^6x^{17} + a^{11}x^{16} + ax^{15} + a^{14}x^{14} + a^7x^{13} + a^{12}x^{12} + a^2x^{11} + a^{12}x^{10} + a^{11}x^9 + a^{14}x^8 + a^{10}x^7 + a^{11}x^6 + a^{14}x^5 + a^5x^4 + a^7x^3 + a^2x^2 + a^9x + a^2),$$

$$g_{23}(x, y) = (a^2x^{25} + a^{14}x^{23} + a^4x^{22} + a^8x^{21} + a^3x^{20} + a^9x^{19} + a^{13}x^{17} + a^{12}x^{16} + a^6x^{15} + a^{12}x^{14} + a^5x^{13} + a^8x^{12} + a^{12}x^{11} + a^6x^{10} + a^{14}x^9 + a^{10}x^8 + ax^7 + x^6 + a^3x^5 + a^{11}x^4 + a^{14}x^3 + a^4x^2 + a^{13}x + a, a^{11}x^{27} + a^6x^{25} + a^4x^{24} + a^{13}x^{23} + a^3x^{22} + a^9x^{21} + a^{13}x^{18} + a^4x^{17} + a^{14}x^{16} + a^3x^{15} + a^8x^{14} + ax^{13} + a^{10}x^{12} + a^7x^{11} + a^9x^{10} + a^{12}x^9 + a^{12}x^8 + a^5x^7 + a^3x^5 + a^{12}x^4 + a^{14}x^2 + a^8x + a^7),$$

$$g_{24}(x, y) = (a^6x^{24} + a^6x^{22} + x^{21} + a^2x^{20} + a^{13}x^{19} + ax^{18} + a^{14}x^{16} + a^{14}x^{15} + a^{11}x^{14} + a^4x^{13} + a^5x^{12} + a^3x^{11} + a^2x^{10} + x^9 + a^9x^8 + a^6x^7 + a^{14}x^6 + a^{11}x^5 + a^6x^4 + a^8x^2 + a^{10}x + a^5, a^{14}x^{26} + a^7x^{25} + x^{24} + a^{11}x^{23} + a^{10}x^{22} + x^{20} + x^{19} + a^{13}x^{17} + a^5x^{16} + a^5x^{15} + a^4x^{14} + a^4x^{13} + a^{11}x^{12} + a^5x^{11} + a^4x^{10} + a^8x^9 + a^{13}x^8 + a^7x^7 + a^{11}x^5 + a^6x^4 + a^8x^3 + a^3x + a^{11}),$$

$$g_{25}(x, y) = [(a^6x^{25} + a^6x^{23} + x^{22} + a^2x^{21} + a^{13}x^{20} + ax^{19} + a^{14}x^{17} + a^{14}x^{16} + a^{11}x^{15} + a^4x^{14} + a^5x^{13} + a^3x^{12} + a^2x^{11} + x^{10} + a^9x^9 + a^6x^8 + a^{14}x^7 + a^{11}x^6 + a^6x^5 + a^8x^3 + a^{10}x^2 + a^5x, a^{14}x^{27} + a^7x^{26} + x^{25} + a^{11}x^{24} + a^{10}x^{23} + x^{21} + x^{20} + a^{13}x^{18} + a^5x^{17} + a^5x^{16} + a^4x^{15} + a^4x^{14} + a^{11}x^{13} + a^5x^{12} + a^4x^{11} + a^8x^{10} + a^{13}x^9 + a^7x^8 + a^{11}x^6 + a^6x^5 + a^8x^4 + a^3x^2 + a^{11}x),$$

$$g_{26}(x, y) = (a^4x^{24} + a^2x^{23} + a^6x^{22} + a^8x^{21} + a^4x^{20} + a^2x^{19} + a^7x^{18} + a^7x^{17} + x^{16} + a^{14}x^{15} + a^{13}x^{14} + ax^{13} + a^6x^{12} + a^{13}x^{11} + a^{11}x^{10} + a^5x^9 + ax^8 + a^9x^7 + a^{12}x^6 + a^9x^4 + ax^3 + a^{13}x^2 + a^{12}x + a^9, a^3x^{28} + a^7x^{27} + a^{11}x^{26} + a^8x^{25} + a^9x^{23} + a^8x^{22} + x^{21} + a^4x^{20} + a^9x^{19} + a^4x^{18} + a^9x^{17} + ax^{16} + a^5x^{15} + a^{13}x^{14} + a^9x^{13} + a^6x^{12} + ax^{11} + a^9x^{10} + a^5x^9 + a^{11}x^8 + a^{10}x^6 + a^5x^5 + a^{10}x^4 + a^{12}x^3 + x^2 + a^8x + 1),$$

$$g_{27}(x, y) = (a^2x^{25} + x^{24} + x^{23} + a^{14}x^{22} + a^6x^{21} + ax^{20} + a^{12}x^{19} + a^{14}x^{18} + a^{10}x^{17} + a^{13}x^{16} + a^5x^{14} + a^5x^{13} + a^{14}x^{12} + a^9x^{11} + a^6x^{10} + a^{13}x^9 + a^{10}x^8 + a^8x^7 + a^3x^6 + a^{12}x^5 + a^7x^4 + x^3 + a^{10}x + a^8, a^6x^{28} + x^{27} + x^{26} + a^4x^{25} + a^6x^{24} + a^4x^{23} + a^{11}x^{21} + a^5x^{20} + a^{11}x^{19} + a^9x^{18} + a^8x^{17} + ax^{16} + a^{13}x^{15} + a^4x^{14} + x^{13} + a^3x^{11} + a^2x^{10} + a^4x^9 + a^5x^8 + x^7 + a^3x^6 + a^{12}x^5 + a^{11}x^4 + a^6x^3 + a^{14}x^2 + a^{13}x + a^{14}).$$

APPENDIX A. MAGMA IMPLEMENTATION OF THE MULTIPLICATION ALGORITHM IN
THE FINITE FIELD $\mathbb{F}_{16^{13}}$ OVER THE FINITE FIELD \mathbb{F}_{16}

```
// Construction of the function field
n:=13; g:=2; q:=16; F16<a>:=GF(16);
Kx<x> := FunctionField(F16);
R<x>:=PolynomialRing(F16);
Kxy<y> := PolynomialRing(Kx);
f:=y^2 + y + x^5;
F<c> := FunctionField(f);
// Construction of the place Q and divisor D
q := x^13+a^6*x^12+a^5*x^11+a^11*x^10+x^9+a^12*x^8+a^7*x^7+
a^7*x^5+a^2*x^4+a^11*x^3+a^8*x^2+a^6*x+a^14;
DD:=x^14+a^9*x^13+a^6*x^12+a^7*x^11+a^11*x^10+a^12*x^9+
a^10*x^8+a^6*x^7+a^7*x^6+a^10*x^5+a^14*x^4+x^3+x^2+a^3*x+a;
P:=Decomposition(F,Zeros(Kx!q)[1]); Q:=P[1];
D:=Decomposition(F,Zeros(Kx!DD)[1])[1]; D:=1D;
IsSpecial(1D); // false
Dimension(Q-D); // 0
IsSpecial(D-Q); // false
// Construction of the residue class field
// and the degree one places
K<b>:=ResidueClassField(Q);
LP:=Places(F,1);

// Construction of the Riemann space
LD, h :=RiemannRochSpace(D);
L2D, h2 :=RiemannRochSpace(2D);
BaseLD:=(h(v))@@h2 : v in Basis(LD)];
Base:=ExtendBasis(BaseLD,L2D);
L2D:=[];
for i in [1..2*n+g-1] do
    L2D:=Append(L2D, h2(Base[i]));
end for;
ML2D:=Matrix(2*n+g-1,1,L2D);
// Construction of E : E=Evalf(Q)
L:=[];
for i in [1..n] do
    L:=Append(L,ElementToSequence(Evaluate(L2D[i],Q)));
end for;
E:=Transpose(Matrix(L));
// we use a normal basis
PP:=[];
for i:=0 to n-1 do
    PP:=Append(PP, ElementToSequence(b^(q^i)));
end for;
NC:=Transpose(Matrix(F16,n,n,PP)); NCI:=NC^-1;
// X.M=fx
M:=Matrix(F,E^-1*NC);
Ev:=Matrix(1,n,[L2D[i] : i in [1..n]]);
// Construction of L(2D) with the required properties
EL2D:=Matrix(2*n+g-1,1,
    [Evaluate(L2D[i],Q) : i in [1..2*n+g-1]]);
```

```

BEL2D:=Matrix(F16,2*n+g-1,n,
  [ElementToSequence(EL2D[i][1]) : i in [1..2*n+g-1]]);
MM:=Parent(ZeroMatrix(F,n+g-1,2*n+g-1))!
  Matrix(Basis(NullSpace(BEL2D)))*ML2D;
for i in [1..n] do
  L2D[i]:=Transpose(EvM)[i,1];
end for;
// rows of ML2D form a basis of L(2D)
ML2D:=Matrix(2*n+g-1,1,
  [L2D[i] : i in [1..n]] cat [MM[i,1] : i in [1..n+g-1]]);
// we can check that evaluation in
// Q of the last n+g-1 gives 0 :
// [Evaluate(ML2D[i,1],Q) : i in [1..2*n+g-1]];

// Construction of T and T^-1
ST:=[];
for j:=1 to 2*n+g-1 do
  for i:=1 to 2*n+g-1 do
    ST:=Append(ST, Evaluate(ML2D[i,1], LP[j]));
  end for;
end for;
T:=Matrix(2*n+g-1, ST);
TI:=T^-1;
// Construction of P
P:=VerticalJoin(HorizontalJoin(ScalarMatrix(F16,n,1),
  ZeroMatrix(F16,n,n+g-1)),
ZeroMatrix(F16,n+g-1,2*n+g-1));
// Matrix T1
T1:=T*P*TI;
// X and Y are the elements to multiply
// represented in a normal basis
X:=Matrix(F16,13,1,[a,1,0,0,0,0,0,0,0,0,0,0,0]); // example
Y:=Matrix(F16,13,1,[1,a,a,0,0,0,0,0,0,0,0,0,0]); // example
fx:=VerticalJoin(X,ZeroMatrix(F16,n+g-1,1));
fy:=VerticalJoin(Y,ZeroMatrix(F16,n+g-1,1));
// u = T(fx)T(fy)
u:= Matrix(2*n+g-1,1,
  [(Tfx)[i][1]*(Tfy)[i][1] : i in [1..2*n+g-1]]);
// fz = MM(P*TI*u)
fz:=Matrix(n,1,[(P*TI*u)[i][1] : i in [1..n]]);
// fz gives X*Y in the normal basis

```

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KEVIN ATIGHEHCHI, AIX-MARSEILLE UNIVERSITÉ, LABORATOIRE D’INFORMATIQUE FONDAMENTALE DE MARSEILLE, CASE 901, F13288 MARSEILLE CEDEX 9 FRANCE
E-mail address: Kevin.Atighehchi@univ-amu.fr

STÉPHANE BALLE, AIX-MARSEILLE UNIVERSITÉ, INSTITUT DE MATHÉMATIQUES DE MARSEILLE CASE 907, F13288 MARSEILLE CEDEX 9 FRANCE
E-mail address: Stephane.Ballet@univ-amu.fr

ALEXIS BONNECAZE, AIX-MARSEILLE UNIVERSITÉ, INSTITUT DE MATHÉMATIQUES DE MARSEILLE CASE 907, F13288 MARSEILLE CEDEX 9 FRANCE
E-mail address: Alexis.Bonnecaze@univ-amu.fr

ROBERT ROLLAND, AIX-MARSEILLE UNIVERSITÉ, INSTITUT DE MATHÉMATIQUES DE MARSEILLE CASE 907, F13288 MARSEILLE CEDEX 9 FRANCE
E-mail address: Robert.Rolland@univ-amu.fr